PROPAGATION OF A SOLITON ALONG A FLUID-FILLED PIPE

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UDC 532.542+539.3

In [1, 2] a mathematical model of the motion of a fluid in a pipe whose axis is a curve in space was discussed and certain simplifications of the problem were studied. The propagation of linear and nonlinear waves in the framework of the model was studied. In the present paper we consider a simple wave flow in a pipe with elastic walls suing one of the models introduced in [1], which, unlike [2], takes into account axial displacements of the pipe. The basic equations describing the propagation of waves in the pipe are obtained.

The equations describing the flow of a fluid in a pipe are written in the form [1]

$$\rho_1 R\frac{\partial^2 u}{\partial t^2} = R\frac{\partial \sigma_{ss}}{\partial s} + R\frac{\sigma_{rs}^e - \sigma_{rs}^i}{h}, \tag{1}$$

$$\rho_1 R\frac{\partial^2 v}{\partial t^2} = -\sigma_{\theta\theta} + R\frac{\sigma_{rr}^e - \sigma_{rr}^i}{h} + R\frac{\partial \sigma_{rs}}{\partial s},$$

$$\sigma_{rs}^i = \mu\frac{u - u^i}{h/2} + \mu\frac{\partial v^i}{\partial s}, \quad \sigma_{rs}^e = \mu\frac{u^e - u}{h/2} + \mu\frac{\partial v^e}{\partial s},$$

$$\sigma_{rs} = \mu\frac{u^e - u^i}{h} + \mu\frac{\partial v}{\partial s}, \quad \sigma_{\theta\theta} = \lambda\frac{\partial u}{\partial s} + (\lambda + 2\mu)\frac{v}{R_0} + \lambda\frac{v^e - v^i}{h},$$

$$\sigma_{ss} = (\lambda + 2\mu)\frac{\partial u}{\partial s} + \lambda\frac{v^e - v^i}{h} + \lambda\frac{v}{R_0},$$

$$\rho V R^2 = \rho_0 V_0 R_0^2, \quad \frac{V^2}{2} + \int_p^{p_0}\frac{dp}{\rho} = \frac{V_0^2}{2}, \quad p - p_0 = a_0^2(\rho - \rho_0).$$

Here u, v are the displacements of the pipe in the longitudinal (s axis) and radial (axis r) directions; the indices e and i refer to quantities evaluated on the outer and inner surfaces of the pipe Φ^e and Φ^i; ρ_1 is the density of the pipe; R = R(s, t) is the generatrix of the pipe; h is its thickness; σ_{ij} is the stress tensor; λ, μ are the Lamé coefficients; t is the time; ρ, V, and p are the density, velocity, and pressure of the fluid (Fig. 1).

Since the components of a vector perpendicular to the generatrix of the pipe satisfy the condition $n_r/n_s \sim (R - R_0)/L \ll 1$, where L is the characteristic linear dimension of the wave, we have

$$\sigma_{rr}^e = \sigma_{rs}^e = \sigma_{rs}^i = 0, \quad \sigma_{rr}^i = -p. \tag{2}$$

Equations (1) and (2) are closed by means of the relations

$$u = (u^e + u^i)/2, \; v = (v^e + v^i)/2, \tag{3}$$

which imply that the differences of the one-sided derivatives with respect to r are continuous in the middle of the surface, and by the geometric relation

$$v - v^0 = R - R_0, \tag{4}$$

where v^0 is the displacement in the unperturbed state at $R = R_0$ due to the pressure p_0.

We note that in the unperturbed state, where $u = u^i = u^e = 0$, it is not difficult to determine the radial deformations v^0, $v_0{}^e$, $v_0{}^i$ and stress:

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Technicheskoi Fiziki, No. 3, pp. 58-63, May-June, 1986. Original article submitted March 13, 1985.

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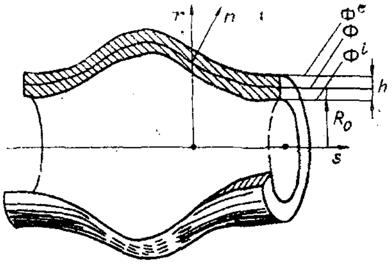

Fig. 1

$$\frac{v^0}{R_0}=\frac{p_0R_0}{4\mu h}\frac{\lambda+2\mu}{\lambda+\mu},\quad \frac{v_0^e-v_0^i}{R_0}=-\frac{p_0\lambda}{4\mu(\lambda+\mu)},\quad \sigma_{ss}=\frac{\lambda}{\lambda+\mu}\frac{p_0R_0}{2h}.$$

This solution reduces to the solution of the Lamé problem to within terms of order $(h/R_0)^2$.

We seek a solution of the system (1) through (4) in the form of a simple wave propagating with velocity D. Then we have the following equations for the functions u and v:

$$(\rho_1D^2-(\lambda+2\mu))\frac{d^2u}{d\xi^2}=\frac{\lambda}{R_0}\frac{dv}{d\xi},\tag{5}$$

$$\rho_1D^2R\frac{d^2v}{d\xi^2}=-\lambda\frac{du}{d\xi}-(\lambda+2\mu)\frac{v}{R_0}-\frac{\lambda\left(v_0^e-v_0^i\right)}{h}+\frac{Rp}{h}$$

($\xi = x - Dt$). Integrating the first equation of (5) with the use of the boundary conditions $u = du/d\xi = 0$, $v = v^0$ when $\xi \to \infty$, and substituting the result into the second equation, we find the following equation for the perturbation $R - R_0$

$$\rho_1D^2R\frac{d^2R-R_0}{d\xi^2}=\frac{(\lambda+2\mu)\rho_1D^2-4\mu(\lambda+\mu)}{\rho_1D^2-(\lambda+2\mu)}\frac{R-R_0}{R_0}+\frac{pR-p_0R_0}{h},$$

which in terms of the dimensionless variables $x = R/R_0$, $\xi = \xi/R_0$ can be written in the form

$$x\ddot{x}=-E(x-1)+\alpha x(p/p_0-1),\tag{6}$$

$$E=\frac{1}{\rho_1D^2}\frac{\rho_1D^2(\lambda+2\mu)-4\mu(\lambda+\mu)}{\rho_1D^2-(\lambda+2\mu)},\quad \alpha=p_0R_0/\rho_1D^2h,$$

which is the same as the equation for the displacement of the walls of the pipe in the presence of the wave as obtained in [2] in the framework of the Zhukovskii model. We note that the first term on the right-hand side of (6) represents the elastic force of the walls of the pipe, which opposes perturbations of the radius when $E > 0$; the second term is the perturbation due to the express pressure $p - p_0$, which arises from the braking of the fluid.

We consider (6) for the case when the velocity D is such that $\rho_1D^2 \ll \lambda+2\mu$, $\rho_1D^2 \ll \ll \frac{4\mu(\lambda+\mu)}{\lambda+2\mu}$. In this case $E\sim\frac{1}{\rho_1D^2}\frac{4\mu(\lambda+\mu)}{\lambda+2\mu}$. . Equation (6) is then written as

$$\ddot{x}=-E(x-1)+\alpha'x(f(y)-1),\tag{7}$$

$$\alpha'=\rho_0a_0^2R_0/\rho_1D^2h,\ y=V/V_0.$$

Here the coefficient of $\ddot{x}$ is "frozen"; this is correct when $|1 - E/A| \ll 1$, where $A = \alpha'\dot{f}(1)\dot{y}(1)$. The function $\rho/\rho_0 = f(y)$ and the dependence of the dimensionless pipe radius on the fluid velocity $x = x_p(y)$ are determined from the equations of motion of the fluid:

$$\rho VR^2=\rho_0V_0R_0^2,$$

$$\frac{V^2}{2}+a_0^2\ln\rho=\frac{V_0^2}{2}+a_0^2\ln\rho_0,$$

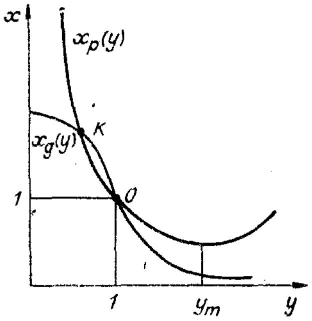

Fig. 2

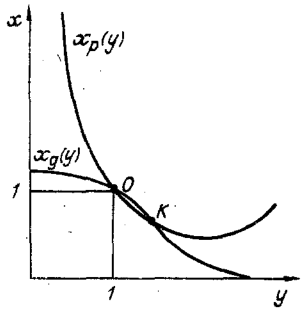

Fig. 3

$$x_p(y) = y^{-\frac{1}{2}} \exp\left(M_0^2 (y^2 - 1)/4\right),$$
$$f(y) = \exp\left(M_0^2 (1 - y^2)/2\right), \quad M_0^2 = V_0^2/a_0^2.$$

We look for equilibrium points for which $\ddot{x} = 0$, $\dot{x} = 0$ (singular points of (7)). The equilibrium points can be found as the points of intersection of the curve $x = x_p(y)$ and the null line $x = x_g(y)$, along which the right hand side of (7) vanishes:

$$x_g(y) = E/(E - \alpha'(f(y) - 1)).$$

We assume that $M_0^2 < 2 \ln \frac{E + \alpha'}{\alpha'}$; then the denominator will be nonzero and the curve $x = x_g(y)$ decreases monotonically. The curve $x = x_p(y)$ has a minimum at the point $y_m = M_0^{-1}$. We assume that $M_0 < 1$, then the intersection point of $x_p(y)$ and $x_g(y)$ lies in the subsonic branch. If $|\dot{x}_g(1)|$ is larger than $|\dot{x}_p(1)|$, then the second intersection point K lies to the left of $y = 1$ (Fig. 2) and in the opposite case it lies to the right of $y = 1$ (Fig. 3). Calculating the derivative, we write these conditions in the form

$$x_K > 1, \quad y_K < 1, \quad M_0^2 \in (M_K^2, 1), \quad M_0^2 < 2 \ln \frac{E + \alpha'}{\alpha'}; \tag{8}$$

$$x_K < 1, \quad y_K > 1, \quad M_0^2 < M_K^2, \quad M_0^2 < 2 \ln \frac{E + \alpha'}{\alpha'}. \tag{9}$$

Here $M_K^2 = (1 + 2\alpha'/E)^{-1}$. . The condition $|1 - E/A| \ll 1$ means that $M_0{}^2$ must be close to $M_K{}^2$. According to the general theory [3] the type of singular point $x = \tilde{x}$ is determined by the sign of $G_{xx}(\tilde{x})$, where G is the potential function:

$$\ddot{x} = -dG/dx. \tag{10}$$

For the case considered here

$$\frac{d^2G}{dx^2} = \frac{E}{x_g} - \alpha' x \frac{df}{dy} \bigg/ \frac{dx_p}{dy}. \tag{11}$$

Expressing f_y in terms of the slope of the curve $x = x_g(y)$ and substituting the result into (11), we obtain at the intersection $x = \tilde{x}$ of $x_p(y)$ and $x_g(y)$:

$$G_{xx}(\tilde{x}) = \frac{E}{\tilde{x}} \left(\frac{d(x_p - x_g)}{dy} \bigg/ \frac{dx_p}{dy} \right).$$

The sign of G_{xx} can be determined from the conditions (8) and (9), if we know the sign of the quantity dx_p/dy and the change of sign of $(x_p - x_g)$ when we pass through the intersection point. It is not difficult to see (Fig. 2) that when (8) is satisfied we have $G_{xx}(1) < 0$, $G_{xx}(x_K) > 0$ and $x = 1$ is the angular point ("saddle point") and $x = x_K$ is the central point.

When (9) is satisfied the inequalities change sign and then $x = 1$ is the central point and $x = x_K$ is the "saddle point" (Fig. 3).

Hence if (8) is satisfied and $M_K < M_0 < 1$, both equilibrium points belong to the subsonic branch of the double-valued function $y(x)$. The choice of this branch yields a unique (to within a constant) potential function $G(x)$. According to [3], in this case in the phase plane $(x, \dot{x})$ there exists a closed integral curve passing through the point $x = 1$, $\dot{x} = 0$ ("saddle point") which goes around the point $x = x_K$, $\dot{x} = 0$ (central point).

On the basis of this analysis we can formulate the following theorem: the solution $x(\xi)$ of (7), defined in the region $(-\infty, \infty)$ and satisfying the stationarity conditions $\ddot{x}$, $\dot{x} \to 0$, $x \to 1$, $\xi \to \pm\infty$, exists in the form of a solitary wave when (8) is satisfied and when the constant in the energy integral is close to be $G(1)$.

In order to obtain an approximate solution of (7) we use the first integral of (10):

$$\dot{x}^2/2 + G(x) = G(1).$$

Then following [2], we expand the potential $G(x)$ around the point $x = 1$ in a Taylor series up to $(x - 1)^3$. Then the second equilibrium point K is given by

$$x_K - 1 = -2G_{xx}(1)/G_{xxx}(1)$$

and the solution is

$$x(\xi) - 1 = -\frac{3G_{xx}(1)}{G_{xxx}(1)} \operatorname{ch}^{-2}\left(\xi \Big/ \sqrt{-\frac{4}{G_{xx}(1)}}\right). \tag{12}$$

Calculating the derivative, we find

$$G_{xx}(1) = -\frac{E\left(M_0^2/M_K^2 - 1\right)}{1 - M_0^2}, \quad G_{xxx}(1) = 2\alpha' M_0^2 \frac{M_0^4 + 3}{\left(1 - M_0^2\right)^3}, \tag{13}$$

$$x_K - 1 = \frac{E}{\alpha'} \frac{\left(M_0^2/M_K^2 - 1\right)\left(1 - M_0^2\right)^2}{M_0^2\left(M_0^4 + 3\right)}.$$

However this expansion is only valid when M_0^2 is not close to unity, because when $M_0^2 \to 1$ $G_{xx}(1), G_{xxx}(1) \to \infty$. This difficulty is overcome as follows. We will calculate x_K as the point of intersection of the curves $x_p(y)$ and $x_g(y)$ by expanding both functions in Taylor series around $y = 1$ up to λ^2, where $\lambda = 1 - y$:

$$x_p = 1 + \frac{1 - M_0^2}{2}\lambda + \frac{3 + M_0^4}{8}\lambda^2 + \ldots, \tag{14}$$

$$x_g = 1 + \frac{\alpha' M_0^2}{E}\lambda + \frac{\alpha'}{2E}\frac{M_0^2}{M_K^2}\left(M_0^2 - M_K^2\right)\lambda^2.$$

When $\alpha'/E \ll 1$ the quadratic term in the expansion of $x_g(\lambda)$ is negligibly small in comparison with the corresponding term in $x_p(\lambda)$. Solving the equation $x_p(\lambda) = x_g(\lambda)$ we find

$$\lambda_K = \frac{4\left(M_0^2/M_K^2 - 1\right)}{M_0^4 + 3}.$$

This expansion is valid up to $M_0^2 = 1$ because

$$\lambda_K\left(M_0^2 = 1\right) = 2\alpha'/E \ll 1.$$

Substituting the result for λ_K into (14) we obtain

$$x_K - 1 = \frac{4\alpha'}{E}\frac{M_0^2}{M_K^2}\frac{M_0^2 - M_K^2}{3 + M_0^2}. \tag{15}$$

Comparing x_K calculated from (13) and (15) we see that they are the same asymptotically when $M_0^2 \to M_K^2$. Before considering the analysis of this approximate solution, we note one feature

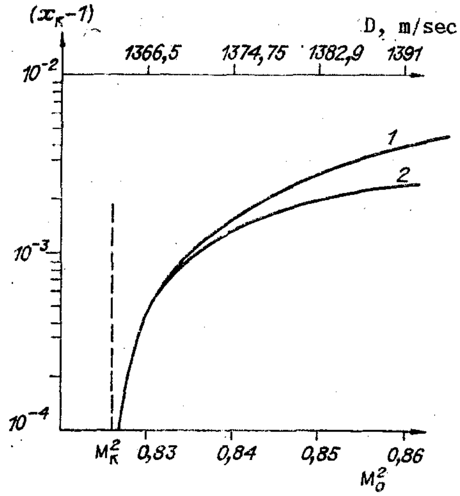

Fig. 4

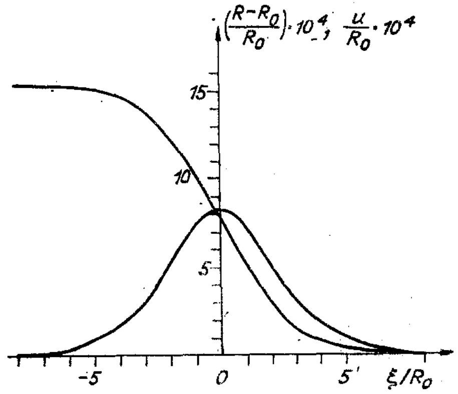

Fig. 5

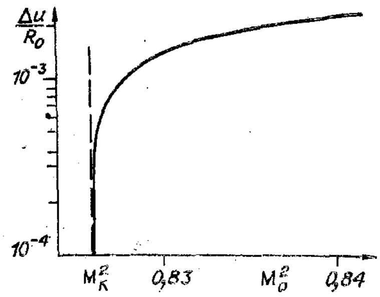

Fig. 6

of this type of channel flow in the quasi-one-dimensional approximation. In [2] the restoring forces were characterized by the Young's modulus E_Y of the material making up the pipe. Here the role of this parameter is played by a quantity which we denote by E (see (6)). When $\lambda = \mu$ (steel), $\rho_1 D^2 << \lambda + 2\mu$ the ratio $(E_Y - E)/E \sim 0.068$. In Fig. 4 we show the dependences predicted by (13) and (15) for $x_K(M_0)$ for a steel pipe filled with water (μ, $\lambda = 8\cdot 10^{10}$ N/m^2, $\rho_1 = 8\cdot 10^3$ kg/m^3, $a_0 = 1.5\cdot 10^3$ m/sec, $\rho_0 = 10^3$ kg/m^3) and with $R_0/h = 10$. Assuming the wave propagates in a stationary medium we have $V_0 = D$; and therefore the velocity scale D of the wave is also shown in Fig. 4. From these graphs it is seen that a difference between (13) and (15) begins to appear only for large deformations of the radius $[(R - R_0)/R_0 \sim 10^{-3}]$, when the deformations cannot be assumed to be elastic. The amplitude of the soliton increases rapidly with D in a narrow region near $a_0/(1 + 2\alpha'/E)$, and (12) is correct up to the elastic limit $x_K - 1 \lesssim 10^{-3}$.

The longitudinal displacement of the middle of the pipe wall is found from (5):

$$\frac{u(\xi)-u(0)}{R_0} = -\frac{\lambda}{\lambda+2\mu}\int_0^{\xi}[x(\tau)-1]\,d\tau = \frac{\lambda}{\lambda+2\mu}\,\frac{3G_{x^2}(1)}{G_{x^3}(1)}\sqrt{\frac{-4}{G_{x^2}(1)}}\;\mathrm{th}\left(\xi\Big/\sqrt{\frac{-4}{G_{x^2}(1)}}\right). \qquad (16)$$

The dependence of the longitudinal and radial displacements (16) and (12) on the coordinate ξ is shown in Fig. 5 for a steel pipe filled with stationary water and with $R_0/h = 10$, $M_0^2 = M_K^2 + 0.005$.

Because of the passage of the soliton, the radius of the pipe takes the value R_0 and in the longitudinal direction the wall is displaced by the quantity $\Delta u/R_0 = \frac{12\lambda}{\lambda+2\mu}\sqrt{-G_{x^2}(1)/G_{x^3}(1)}$. We note that in [2] only radial displacements were associated with the wave. The dependence of $\Delta u/R_0$ on M_0^2 for the same parameters as above is shown in Fig. 6. We see that the longitudinal displacements are larger than the radial ones. This implies that in considering the propagation of stress and deformation waves in pipes one must take into account the quasi-two-dimensional nature of the deformed state of the walls.

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GENERATION OF INTERNAL WAVES UNDER THE COMBINED TRANSLATIONAL AND VIBRATIONAL MOTION OF A CYLINDER IN A FLUID BILAYER

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UDC 532.593

The analysis of internal waves in an inviscid fluid bilayer has been considered in the linear theory for a general form of the motion of the source (see, e.g., [1]). For the special case of the motion of a circular cylinder perpendicular to its generatrix, one of the interesting regimes occurs when the cylinder, translating parallel to the surface, simultaneously performs vertical harmonic oscillations. As shown in [1], the wave field in this case depends in an essential way on the oscillation frequency Ω. For relatively small frequencies waves are excited both in front of and behind the body. When the frequency increases above a certain critical value Ω_* (which depends on the translational velocity of the body, the thicknesses of the fluid layers, and the density difference between them) wave motion is only possible behind the body. When $\Omega = \Omega_*$, the linear theory of an ideal fluid predicts an unbounded growth (as a power law) of the wave amplitude in time, as occurs in resonance phenomena of various kinds. The growth of the wave can be bounded either by viscosity or by nonlinear effects. The effect of viscosity was considered in [2] for a similar plane problem involving excitations created by a horizontally oscillating cylinder moving in a lower layer of an infinite fluid bilayer. In the problem considered in [2], it was assumed that the viscosity was nonzero only in the upper layer. Nonlinear effects have been analyzed in [3, 4], where for the special case of a uniform fluid, nonlinear boundary conditions on the free surface were taken into account. The behavior of internal waves in a linearly stratified fluid has been studied theoretically and experimentally for various types of the motion of the body (see, for example, [5]). The formulation of the problem closest to the one considered here is that of [6].

In the theoretical part of the present paper we are concerned mainly with taking into account viscosity in the framework of the linear model. We also performed experiments in which the critical and near-critical regimes were studied. The present paper is a continuation of [7], where theoretical and experimental results were presented for internal waves generated by the vertical harmonic oscillations of a submerged cylinder in a bilinear of viscous fluid with surface tension at the boundary.

In the theoretical solution of the linear problem for the behavior of internal waves generated by a moving circular cylinder, the cylinder is modelled as a point dipole. The fluid is assumed to be incompressible, is at rest in the unperturbed state, and consists of two infinitely deep layers with small viscosities; the density of the fluid in the upper $(y > 0)$ and lower $(y < 0)$ layers are ρ_1 and $\rho_2 = \rho_1(1 + \varepsilon)(\varepsilon > 0)$, respectively, and the dynamical coefficients of viscosity are μ_1 and μ_2. The surface tension on the boundary

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki Tekhnicheskoi Fiziki, No. 3, pp. 63-70, May-June, 1986. Original article submitted April 25, 1985.

0021-8944/86/2703-0375$12.50